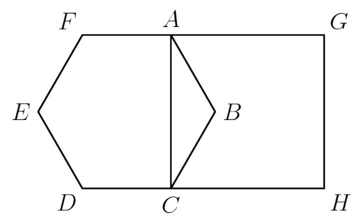
1. How many distinct prime factors does $4^8 - 81$ have?

   (A) 2 (B) 3 (C) 4 (D) 5 (E) 6

2. How many distinct scalene triangles have integer side lengths and a perimeter of 22, up to congruence?

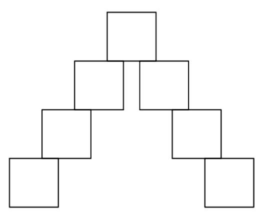
   (A) 5 (B) 6 (C) 8 (D) 10 (E) 11

3. Regular hexagon $ABCDEF$ and square $AGHC$ are shown below. If the length of segment $AF$ is 4, then what is the area of pentagon $AGHCB$?




   (A) $32 - 4\sqrt{3}$ (B) $32 - 2\sqrt{3}$ (C) $48 - 8\sqrt{3}$ (D) $48 - 4\sqrt{3}$ (E) 48

4. Kirk glues seven identical cubes together to form a stack, the front view of which is shown below. With the exception of the top cube, 1/3 of the top face of each cube is covered by the cube above it. Kirk submerges his stack of cubes in red paint. What is the ratio of the surface area of the stack that is painted red to the surface area of the top cube that is painted red?

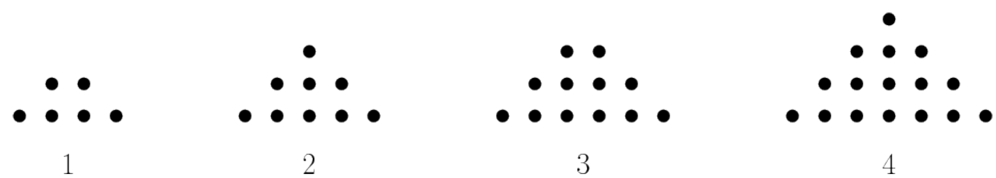


   (A) 57 : 8 (B) 15 : 2 (C) 34 : 5 (D) 20 : 3 (E) 6 : 1

5. The median of the first six numbers in an arithmetic sequence is 20. The median of the first nine numbers in the same sequence is 44. What is the first number in the sequence?

   (A) $-32$ (B) $-20$ (C) $-9$ (D) 0 (E) 3

6. Thisbe is drawing a pattern of dot pyramids. The first four dot pyramids are shown below. If the pattern continues, how many dots will be in dot pyramid 51?




(A) 702 (B) 756 (C) 784 (D) 812 (E) 1312

7. A solid regular octahedron has volume $\sqrt{6}$. What is the surface area of the octahedron?

(A) $\frac{\sqrt{2}}{2}$ (B) $\frac{2\sqrt{2}}{3}$ (C) $2\sqrt{3}$ (D) $4\sqrt{3}$ (E) $6\sqrt{3}$

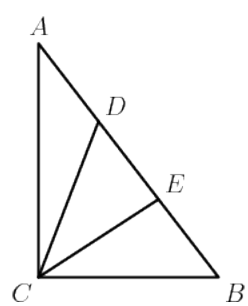
8. License plates in Arstotzka consist of four characters. Each character is either a digit (0-9) or a letter (A, B, or C). The same character may not appear more than once on a plate, and three numerical digits cannot appear in a row. How many unique license plates can be made?

(A) 4320 (B) 4920 (C) 7800 (D) 7860 (E) 12120

9. Positive integers $a, b$, and $c$, with $a \geq 3$ and $b \geq 3$, satisfy $\frac{1}{a} + \frac{1}{b} = \frac{1}{c} + \frac{1}{2}$. Find the number of possible values of $c$.

(A) 2 (B) 3 (C) 4 (D) 5 (E) more than 5

10. In the right triangle $ABC$ shown, $D$ and $E$ lie on the hypotenuse $AB$ so that $AD = DE = EB$ and $CD = 7$ and $CE = 6$. What is the length of hypotenuse $AB$?




(A) $\frac{37}{3}$ (B) $3\sqrt{13}$ (C) $3\sqrt{17}$ (D) $9 + \sqrt{13}$ (E) 13

11. On a windless day, a pegasus can fly from Manehattan to Trottingham and back in 3 hours and 45 minutes. However, when there is a 10 mile per hour wind blowing from Manehattan to Trottingham, it takes the pegasus 4 hours to make the round trip. How many miles is it from Manehattan to Trottingham?

    (A) 40 (B) 75 (C) 150 (D) 300 (E) 600

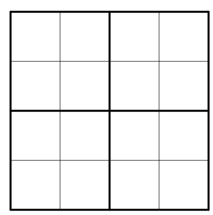
12. Find the largest integer $k$ such that, for any two-digit number $x$, $x^2$ is greater than or equal to $k$ times the product of the digits of $x$.

    (A) 39 (B) 40 (C) 41 (D) 42 (E) 45

13. How many ways are there to fill in the following 4-by-4 grid according to the following rules:

    - Each cell contains exactly one of the integers 1 through 4, inclusive.
    - Each of the numbers 1 through 4, inclusive, is contained in each of the following: the four rows, the four columns, and the four bold 2-by-2 zones.

    (A) 192 (B) 288 (C) 384 (D) 576 (E) 768



14. Ultimate frisbee teams A and B are not evenly matched. Whenever they play, Team A wins with probability 0.6 and team B wins with probability 0.4. Suppose that teams A and B play a series of games until one team has won two more games than the other. What is the probability that team B wins the series?

    (A) $\frac{4}{25}$ (B) $\frac{3}{10}$ (C) $\frac{4}{13}$ (D) $\frac{6}{19}$ (E) $\frac{2}{5}$

15. Suppose the polynomial $p(x) = x^3 - ax^2 + bx - c$ factors as $(x-a)(x-b)(x-c)$, where $a, b$, and $c$ are non-zero. What is the value of $p(2)$?

    (A) $-3$ (B) 0 (C) 4 (D) 7 (E) 9

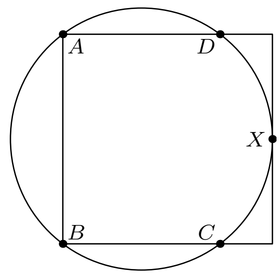
San Diego Math League High School Division, Round 1a
November 16, 2013

Answers

1. C, 4
2. A, 5
3. D, $48 - 4\sqrt{3}$
4. A, $57 : 8$
5. B, $-20$
6. B, 756
7. E, $6\sqrt{3}$
8. C, 7800
9. B, 3
10. C, $3\sqrt{17}$
11. B, 75
12. B, 40
13. B, 288
14. C, $4/13$
15. E, 9

1. The probability of drawing a red marble from a bag is 3/5. After some red marbles are removed, the probability of drawing a red marble is 2/7. What is the smallest number of marbles that could have originally been in the bag?

2. The circle and square in the figure intersect at points $A, B, C, D$, and $X$. Find the ratio of the side of the square to the radius of the circle.




3. The base-5 number $32_5$ is equal to the base-7 number $23_7$. There are two 3-digit numbers in base 5 that similarly have their digits reversed when expressed in base 7. What is their sum, in base 5?

4. What is the largest value of $A$ such that every triangle with area $A$ units squared can be placed between two parallel lines 1 unit apart?

5. For how many positive integers $n < 150$ does there exist a positive integer $m$ such that there are exactly 8 ordered pairs of integers $(x, y)$ for which $\gcd(x, y) = m$ and $\text{lcm}(x, y) = n$?

6. $a, b$, and $c$ are positive integers such that

$$a + bc = 78$$
$$b + ac = 57$$
$$c + ab < 100$$

Find ordered triple $(a, b, c)$.

7. Let $S(k) = \binom{4k}{0} + \binom{4k}{4} + \binom{4k}{8} + \cdots + \binom{4k}{4k}$, where $k$ is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!}$. What is the smallest $k$ such that $S(k)$ is divisible by 7?

8. Inside a regular octagon with side length 1, we color red each point that is within 1 unit of exactly one vertex and color white the remaining points. What is the total area shaded red?

Answers

1. 25
2. 8/5 or 8 : 5
3. 1103 or $1103_5$
4. $\sqrt{3}/3$
5. 18
6. $(6, 9, 8)$
7. 5
8. $4\sqrt{3} - 4\sqrt{2} - \pi/3$